PERGAMON

International Journal of Solids and Structures 38 (2001) 8503–8527

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Plastic continuum with microstructure, local second gradient theories for geomaterials: localization studies

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Received 14 July 1999

Abstract

Many plastic second gradient models have been developed in the last 10 years. However some plastic second gradient models are nonlocal ones. This paper is an attempt to give a general framework to deal with local second gradient theories within theories with microstructure, keeping in mind future applications for geomaterials. It is advocated that particular elasto-plastic local models with microstructure, namely local second gradient and Cosserat second gradient models, which are the least developed in the literature have some advantages which are somewhat promising. One main objective of this paper is to present these two families of models. The first one (Cosserat second gradient model) is shown to be well adapted to granular materials. The second model family is likely to be a good model for cohesive geomaterials. Another aim of this work is to give a method to obtain basic solutions, which can be seen as localization analysis, in one and two dimensions cases. The key point of this method is the use of patch conditions between loading and unloading parts. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Plasticity; Microstructure; Second gradient; Localization

1. Introduction

The second gradient models have often been used since the pioneering work of Aifantis (1984) and Zbib and Aifantis (1988a,b). In most cases, the second gradient approach is treated within the flow theory of plasticity. In this context, elasto-plastic second gradient models have been developed, involving the second gradient of the plastic strain in the consistency condition and/or the flow rule (see e.g. Vardoulakis and Aifantis (1991), de Borst and Muhlhaus (1992a,b), Muhlhaus and Aifantis (1991), Pamin (1994)). Such models can be called nonlocal second gradient plasticity models, as the constitutive equation in its incremental form is itself a partial differential equation.

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[1] Financial support during one-year stay in France from Research Fellowship of the Japan Society for the Promotion of Science for Young Scientists is gratefully acknowledged.

PII: S0020-7683(01)00057-9

The approaches of Chambon et al. (1996) (see also Chambon et al. (1998) and Matsushima et al. (2000)) and Fleck and Hutchinson (1998) (see also Fleck and Hutchinson (1997)) are different. The first one which is developed in the following for two dimensions, is the straight generalization of the classical flow theory of plasticity. The second one is both a generalization of the deformation theory of plasticity and of the flow theory of plasticity (see Fleck and Hutchinson (1998)). Some developments, similar to the ones presented here (see Fleck and Hutchinson (1997)), have been made for modeling metals. However the main difference arises from the motivations. The Fleck–Hutchinson theory is intended to model metal plasticity whereas we want to model geomaterials. Classical (without second gradient effects) metal plasticity is based on firm microscale studies. Such studies has been recently extended with some adding assumptions. This provides a mechanism-based gradient plasticity theory (see e.g. Gao et al. (1999) and Huang et al. (2000)) which supports the initially phenomenological Fleck–Hutchinson theory. The state of affairs is quite different for geomaterials. Even the classical geomaterials plasticity theory cannot be clearly deduced from a microscale study. On the other hand, as rupture is almost always localized and as localization exhibits clearly internal length, it is necessary to explore the possibilities of enhanced models. The only possible way is a study of phenomenological theories. The granular nature of some geomaterials and the clear difference between the strain of the grains and the strain of the material itself (see sand behavior for instance) suggests that the whole family of models with microstructures has to be studied. So in the following more general theories (than second gradient plasticity theories) are presented and complete solutions for problems involving these general theories are given. In some cases however (for granular materials see Section 4.3), some micromechanic experiments suggest that one of these models is better.

The problems solved are mainly related to localization and the method used is a generalization of the one given in Chambon et al. (1998).

We start first from the general theory of media with microstructure following mainly the work of Germain (see e.g. Germain (1973a,b). Other main references in this first part are the work of Mindlin (see e.g. Mindlin (1964, 1965)) and the book by Vardoulakis and Sulem (1995). Natural generalizations of the flow theory of plasticity for media with microstructure are then proposed in the second part. In a third part, we study the links between this theory and the Cosserat theories. A fourth part gives a general framework for local second gradient plasticity models. Here local theory means that the constitutive equation is a relation only between local quantities. For these three theories (namely models with microstructure, Cosserat and second gradient models) which use generalizations of the theory developed in Chambon et al. (1996) and Chambon et al. (1998), one-dimensional solutions of the rate boundary value problems are given. In the fifth part, some local second gradient models are developed in the two-dimensional case and a Mohr–Coulomb model is presented. In a last part, analytical two-dimensional solutions, which can be seen as localization studies for such models, are given. Finally, applications for the Mohr–Coulomb model are computed. It can be seen that different length scales are involved in these solutions.

It is necessary to specify what the main assumptions of this paper are. First, we restrict our study to the so-called small strain assumption. This means that the different configurations of a continuum body are assumed to be identical. Second for the sake of simplicity, we do not consider couple body forces, but only classical ones. The third main restriction of this study is that we deal only with quasi-static problems. This means that we neglect the inertia terms and the so-called microinertia effects as well. The boundaries of studied bodies are assumed to be smooth enough to define one and only one normal for every point.

Let us finally give the principles of our notations. A component is denoted by the name of the tensor (or vector) accompanied by tensorial indices. All tensorial indices are in lower position, as there is no need in the following of a distinction between covariant and contravariant components. Upper indices have other meanings. The summation convention with respect to repeated tensorial indices is used. In order to avoid confusions, squares are systematically denoted with parentheses.

2. A general theory for continua with microstructure

2.1. Some kinematic preliminaries

Kinematics of a classical continuum is defined by a displacement field denoted u_i, function of the coordinates denoted x_i. For some media involving grains or crystals (such as for instance soils and rocks), such a description is sufficient in numerous applications. On the other hand, it is well known that some experimental results, especially those exhibiting a clear scale effect (such as localization phenomena) cannot be predicted by classical modeling. It is then reasonable to add to the previous description a field of second order tensors which models the strains and the rotation of the grains themselves. This field, denoted f_{ij} is nonsymmetric and has nothing to do with the gradient of an underlying displacement field, it is called here microkinematic gradient. In classical models the gradient of the displacement is used to define the internal virtual work which is a linear form of the displacement gradient. In the case of media with microstructure, it is consistent to consider the virtual work as a linear form with respect to the displacement gradient, the microkinematic gradient f_{ij} and its gradient denoted h_{ijk} in the following. We now summarize the previous assumption in the following list of notations:

- u_i is the (macro) displacement field
- F_{ij} is the macro displacement gradient which means:

$$F_{ij} = \frac{\partial u_i}{\partial x_j} \tag{1}$$

- D_{ij} is the macro strain:

$$D_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) \tag{2}$$

- R_{ij} is the macro rotation:

$$R_{ij} = \frac{1}{2}(F_{ij} - F_{ji}) \tag{3}$$

- f_{ij} is the microkinematic gradient. Let us emphasize that it has not to meet compatibility conditions.
- d_{ij} is the microstrain:

$$d_{ij} = \frac{1}{2}(f_{ij} + f_{ji}) \tag{4}$$

- r_{ij} is the microrotation:

$$r_{ij} = \frac{1}{2}(f_{ij} - f_{ji}) \tag{5}$$

- h_{ijk} is the (micro) second gradient:

$$h_{ijk} = \frac{\partial f_{ij}}{\partial x_k} \tag{6}$$

2.2. The internal virtual work

In order to define the internal virtual work it is necessary to assume that virtual variables corresponding to the previous kinematic variables, can be defined and that dual static variables can be defined too. The principle of material frame indifference states that the same virtual work has to be retrieved with kinematics defined with respect to two different frames each one having a solid body motion with respect to the other. Thus the virtual work has to depend only on the macrostrain, the relative deformation gradient (i.e. the difference between the macrodisplacement gradient and the microkinematic gradient) and the (micro) second gradient (see e.g. Germain (1973b)). Denoting with a * virtual quantities, the density (per unit volume) of internal virtual work can be written:

$$w^* = \sigma_{ij} D^*_{ij} + \tau_{ij}(f^*_{ij} - F^*_{ij}) + \chi_{ijk} h^*_{ijk} \tag{7}$$

σ_{ij} is called here the macro stress. τ_{ij} is an additive stress associated with the microstructure, it is not necessarily symmetric and is called microstress. χ_{ijk} which is related with h^*_{ijk} is called the double stress. We have to be careful here that D^*_{ij} and F^*_{ij} are depending on a virtual displacement field u^*_i by the way of equations similar to Eqs. (1) and (2). Similarly h^*_{ijk} is depending on f^*_{ij} by an equation similar to Eq. (6). Finally the internal virtual work for a given body Ω reads:

$$W^{*\mathrm{i}} = \int_\Omega w^* \,\mathrm{d}v = \int_\Omega (\sigma_{ij} D^*_{ij} + \tau_{ij}(f^*_{ij} - F^*_{ij}) + \chi_{ijk} h^*_{ijk}) \,\mathrm{d}v \tag{8}$$

Other decompositions of the virtual work are possible as it will be seen for instance in Section 4.

2.3. The external virtual work

In order to be able to get the balance equations it is necessary to make some assumptions about the external virtual work. This means that we introduce here the external forces. We do not want to give the more general case, so we assume that only classical body forces (denoted G_i) are applied, this means precisely that there is no body double force (see e.g. Germain (1973b) for more general assumptions). On the other hand, we assume that not only the classical traction forces t_i but also double surface tractions T_{ij} are acting on the boundary. Finally, denoting as usual $\partial\Omega$ the boundary of Ω, the external virtual work reads:

$$W^{*\mathrm{e}} = \int_\Omega G_i u^*_i \,\mathrm{d}v + \int_{\partial\Omega} (t_i u^*_i + T_{ij} f^*_{ij}) \,\mathrm{d}s \tag{9}$$

G_i is assumed to be known in every point of Ω, t_i and similarly T_{ij} are assumed to be known at least on a part of $\partial\Omega$.

2.4. The balance equations and the boundary conditions

By equating the external virtual work (Eq. (9)) and the internal virtual work (Eq. (8)) for all kinematic admissible virtual fields, we get first the balance equations and second the boundary conditions (the one involving the given external forces). A kinematic admissible virtual field is a field which is sufficiently smooth and which has a null value on the part of the boundary where the corresponding real field is prescribed. At this step let us emphasize that there is no link between u^*_i an f^*_{ij}. We do not present here the calculations which are classical and based on one integration by parts and the divergence formula. So we get the balance equations:

$$\frac{\partial(\sigma_{ij} - \tau_{ij})}{\partial x_j} + G_i = 0 \tag{10}$$

$$\frac{\partial \chi_{ijk}}{\partial x_k} - \tau_{ij} = 0 \tag{11}$$

and the static boundary conditions:

$$(\sigma_{ij} - \tau_{ij}) n_j = t_i \tag{12}$$

$$\chi_{ijk} n_k = T_{ij} \tag{13}$$

where n_j is the external normal to the boundary $\partial\Omega$. In order to get a complete problem, it is necessary to prescribe on the boundary $\partial\Omega$ either values of t_i and T_{ij} or values of u_i and f_{ij}. Like in classical media it is possible to assumed that for some part of $\partial\Omega$ some mixed boundary conditions are prescribed.

3. Flow theories of plasticity for microstructured continuum

In order to make our problem complete, it is necessary to specify the constitutive equation. Let us denote now $fF_{ij} = f_{ij} - F_{ij}$. It is natural to generalize the classical way of building constitutive equations given by Truesdel and Noll (1965). The generalized stresses σ_{ij}, τ_{ij} and χ_{ijk} denoted generically Σ are known in every point of the material if the history of the generalized strains D_{ij}, fF_{ij} and h_{ijk} denoted generically E is known at the same point. The history of the whole kinematics gives the set of different stresses.

$$\Sigma(t^f) = \Upsilon(E(t), t \in [0, t^f]) \tag{14}$$

where t is the time and t^f a given time. This equation defines a local continuum because the generalized stress depend only on the local kinematics history.

We can first consider a hyperelastic model by defining a potential Π from which Σ is deriving.

$$\Sigma = \frac{\mathrm{d}\Pi}{\mathrm{d}E} \tag{15}$$

If we assume moreover that this potential is quadratic, then the generalized stress Σ is linear with respect to the generalized strain E. A possible choice is the following.

$$\Pi = \tfrac{1}{2}\{(D_{ij}\Gamma^1_{ijkl}D_{kl} + fF_{ij}\Gamma^2_{ijkl}(fF_{kl}) + h_{ijk}\Gamma^3_{ijklmn}h_{lmn})\} \tag{16}$$

where Γ^1_{ijkl}, Γ^2_{ijkl} and Γ^3_{ijkl} are symmetric tensors (this choice is not the most general one). Then:

$$\sigma_{ij} = \Gamma^1_{ijkl}D_{kl} \tag{17}$$

$$\tau_{ij} = \Gamma^2_{ijkl}fF_{kl} \tag{18}$$

$$\chi_{ijk} = \Gamma^3_{ijklmn}h_{lmn} \tag{19}$$

3.1. A one-mechanism flow theory of plasticity

It is straightforward to develop an isotropic elasto-plastic model involving one plastic mechanism, as a generalization of the classical flow theory of plasticity. The generalized strain rate is split in an elastic part and a plastic one.

$$\dot{E} = \dot{E}^{\mathrm{e}} + \dot{E}^{\mathrm{p}} \tag{20}$$

The elastic part obeys for instance the previous elastic model.

$$\dot{\sigma}_{ij} = \Gamma^1_{ijkl}\dot{D}^{\mathrm{e}}_{kl} \tag{21}$$

$$\dot{\tau}_{ij} = \Gamma^2_{ijkl}\dot{f}F^{\mathrm{e}}_{kl} \tag{22}$$

$$\dot{\chi}_{ijk} = \Gamma^3_{ijklmn}\dot{h}^{\mathrm{e}}_{lmn} \tag{23}$$

Let $\phi(\sigma_{ij}, \tau_{ij}, \chi_{ijk}, \kappa)$ be the yield function, where κ is a hardening parameter. It has to remain negative or null $\phi(\sigma_{ij}, \tau_{ij}, \chi_{ijk}, \kappa) \leqslant 0$.

In the elastic zone, if $\phi(\sigma_{ij}, \tau_{ij}, \chi_{ijk}, \kappa) < 0$ or for unloading conditions, if $\phi(\sigma_{ij}, \tau_{ij}, \chi_{ijk}, \kappa) = 0$ and $\dot{\phi}(\sigma_{ij}, \tau_{ij}, \chi_{ijk}, \kappa) < 0$ then the strain rate is only elastic $\dot{E} = \dot{E}^{\mathrm{e}}$ and Eqs. (24)–(26) give the evolution of the stresses.

$$\dot{\sigma}_{ij} = \Gamma^1_{ijkl}\dot{D}_{kl} \tag{24}$$

$$\dot{\tau}_{ij} = \Gamma^2_{ijkl}\dot{f}F_{kl} \tag{25}$$

$$\dot{\chi}_{ijk} = \Gamma^3_{ijklmn}\dot{h}_{lmn} \tag{26}$$

Otherwise, for unloading condition $\phi(\sigma_{ij}, \tau_{ij}, \chi_{ijk}, \kappa) = 0$ and $\dot{\phi}(\sigma_{ij}, \tau_{ij}, \chi_{ijk}, \kappa) = 0$, it is assumed that the direction of the plastic strain rate is known:

$$\dot{D}^{\mathrm{p}}_{ij} = \lambda\psi^D_{ij} \tag{27}$$

$$\dot{f}F^{\mathrm{p}}_{ij} = \lambda\psi^{fF}_{ij} \tag{28}$$

$$\dot{h}^{\mathrm{p}}_{ijk} = \lambda\psi^h_{ijk} \tag{29}$$

where λ is the plastic multiplier and ψ^D_{ij}, ψ^f_{ij} and ψ^h_{ijk} are the directions of the plastic strains which are given functions of the state (the stress and the hardening parameter). The consistency condition (Eq. (30)) and the hardening rule which gives the relation between κ and for instance the plastic strain allow to compute the stress rate for any strain rate when plasticity takes place.

$$\frac{\partial\phi}{\partial\sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial\phi}{\partial\tau_{ij}}\dot{\tau}_{ij} + \frac{\partial\phi}{\partial\chi_{ijk}}\dot{\chi}_{ijk} + \frac{\partial\phi}{\partial\kappa}\dot{\kappa} = 0 \tag{30}$$

Let us denote as usual h the hardening modulus

$$\frac{\partial\phi}{\partial\kappa}\dot{\kappa} = -\lambda h \tag{31}$$

and defining H as

$$H = h + \frac{\partial\phi}{\partial\sigma_{ij}}\Gamma^1_{ijkl}\psi^D_{kl} + \frac{\partial\phi}{\partial\tau_{ij}}\Gamma^2_{ijkl}\psi^{fF}_{kl} + \frac{\partial\phi}{\partial\chi_{ijk}}\Gamma^3_{ijklmn}\psi^h_{lmn} \tag{32}$$

If loading occurs, then we have

$$\dot{\sigma}_{ij} = \Gamma^1_{ijkl}\dot{D}_{kl} - \frac{1}{H}\Gamma^1_{ijpq}\psi^D_{pq}\frac{\partial\phi}{\partial\sigma_{mn}}\Gamma^1_{mnkl}\dot{D}_{kl} - \frac{1}{H}\Gamma^1_{ijpq}\psi^D_{pq}\frac{\partial\phi}{\partial\tau_{mn}}\Gamma^2_{mnkl}\dot{f}F_{kl} - \frac{1}{H}\Gamma^1_{ijpq}\psi^D_{pq}\frac{\partial\phi}{\partial\chi_{krs}}\Gamma^3_{krslmn}\dot{h}_{lmn} \tag{33}$$

$$\dot{\tau}_{ij} = \Gamma^2_{ijkl}\dot{f}F_{kl} - \frac{1}{H}\Gamma^2_{ijpq}\psi^{fF}_{pq}\frac{\partial\phi}{\partial\sigma_{mn}}\Gamma^1_{mnkl}\dot{D}_{kl} - \frac{1}{H}\Gamma^2_{ijpq}\psi^{fF}_{pq}\frac{\partial\phi}{\partial\tau_{mn}}\Gamma^2_{mnkl}\dot{f}F_{kl} - \frac{1}{H}\Gamma^2_{ijpq}\psi^{fF}_{pq}\frac{\partial\phi}{\partial\chi_{krs}}\Gamma^3_{krslmn}\dot{h}_{lmn} \tag{34}$$

$$\dot{\chi}_{ijk} = \Gamma^3_{ijklmn}\dot{h}_{lmn} - \frac{1}{H}\Gamma^3_{ijkpqr}\psi^h_{pqr}\frac{\partial\phi}{\partial\sigma_{mn}}\Gamma^1_{mnkl}\dot{D}_{kl} - \frac{1}{H}\Gamma^3_{ijkpqr}\psi^h_{pqr}\frac{\partial\phi}{\partial\tau_{mn}}\Gamma^2_{mnkl}\dot{f}F_{kl} - \frac{1}{H}\Gamma^3_{ijkpqr}\psi^h_{pqr}\frac{\partial\phi}{\partial\chi_{krs}}\Gamma^3_{krslmn}\dot{h}_{lmn} \tag{35}$$

Eqs. (33)–(35) show clearly that in the case of one mechanism, even without any elastic coupling between kinematic variables, the model exhibits a coupling due to its plastic part. For the plastic branch, the macrostress rate depends not only on the macrostrain rate but also on the microkinematic gradient rate and on the second gradient rate. Similar observations are available for the microstress rate and the double stress rate.

3.2. A multi-mechanism plastic theory

It is straightforward to develop an elasto-plastic theory with several mechanisms. In particular it is interesting to consider a model with three mechanisms, each one corresponding, respectively, to the macrostress, the microstress and the (micro) double stress. We have to deal with three plastic multipliers. In geomechanics for instance, physical reason may lead to do so, the microstress would be linked to grain straining and macrostress to rearrangement of grains. Assuming that these three mechanisms are independent on each other, we define three yield functions $\phi^\sigma(\sigma_{ij}, \kappa^\sigma)$, $\phi^\tau(\tau_{ij}, \kappa^\tau)$ and $\phi^\chi(\chi_{ijk}, \kappa^\chi)$ and three hardening parameters κ^σ, κ^τ and κ^χ corresponding to macrostrain, microstrain and microstrain gradient, respectively. As the three mechanisms are supposed to be independent, we get three plastic multipliers denoted λ^σ, λ^τ and λ^χ, respectively, and then according to the loading criterion of each mechanism, either we get relations similar to Eqs. (24)–(26), or we get the following Eqs. (36)–(38) according to which mechanisms are loading.

$$\dot{\sigma}_{ij} = \Gamma^1_{ijkl}\dot{D}_{kl} - \frac{\Gamma^1_{ijpq}\psi^D_{pq}\frac{\partial\phi^\sigma}{\partial\sigma_{mn}}\Gamma^1_{mnkl}\dot{D}_{kl}}{h^\sigma + \frac{\partial\phi}{\partial\sigma_{mn}}\Gamma^1_{mnkl}\psi^D_{kl}} \tag{36}$$

$$\dot{\tau}_{ij} = \Gamma^2_{ijkl}\dot{f}F_{kl} - \frac{\Gamma^2_{ijpq}\psi^{fF}_{pq}\frac{\partial\phi^\tau}{\partial\tau_{mn}}\Gamma^2_{mnkl}\dot{f}F_{kl}}{h^\tau + \frac{\partial\phi}{\partial\tau_{mn}}\Gamma^2_{mnkl}\psi^{fF}_{kl}} \tag{37}$$

$$\dot{\chi}_{ijk} = \Gamma^3_{ijklmn}\dot{h}_{lmn} - \frac{\Gamma^3_{ijkpqr}\psi^h_{pqr}\frac{\partial\phi^\chi}{\partial\chi_{qrs}}\Gamma^3_{qrslmn}\dot{h}_{lmn}}{h^\chi + \frac{\partial\phi}{\partial\chi_{qrs}}\Gamma^3_{qrstuv}\psi^h_{tuv}} \tag{38}$$

where h^σ, h^τ and h^χ are the three hardening modulus corresponding respectively to the three mechanisms. Obviously many other possibilities are available to built up a multi-mechanism plasticity model for media with microstructure. A multi-mechanism theory with coupled mechanisms is easy to study too but for brevity we do not want to detail it here.

3.3. Energetic consequences

The advantages of flow theories of plasticity are well known, and we retrieve them for these theories with microstructure. From a thermodynamic viewpoint, the variation of the internal mechanical energy $\dot{\Pi}$ and the energy dissipation $\dot{\Theta}$ are:

$$\dot{\Pi} = \sigma_{ij}\dot{D}^{\mathrm{e}}_{ij} + \tau_{ij}\dot{f}F^{\mathrm{e}}_{ij} + \chi_{ijk}\dot{h}^{\mathrm{e}}_{ijk} \tag{39}$$

$$\dot{\Theta} = \sigma_{ij}\dot{D}^{\mathrm{p}}_{ij} + \tau_{ij}\dot{f}F^{\mathrm{p}}_{ij} + \chi_{ijk}\dot{h}^{\mathrm{p}}_{ijk} \tag{40}$$

The energy dissipation $\dot{\Theta}$ has to be negative or equal to zero. The sum of these two quantities is equal to the external mechanical power supply, it is then easy to check the thermodynamics consistency of this simple theory.

3.4. One-dimensional applications

3.4.1. Assumptions

In order to have some insight into the behavior of the models presented above, let us study a one-dimensional problem. Denoting x the space variable, we assume that displacements denoted u are measured along the same axis as x. The derivative with respect to x is denoted by a $'$. The macro strain has a unique component denoted u'. The micro kinematic field has only one component denoted f. It corresponds to a uniaxial (micro) straining along the same axis. The micro (second) gradient is then denoted f'. Each of the macrostress, the microstress and the micro double stress has a unique component denoted, respectively, σ, τ and χ. The body forces are assumed to remain constant, so their time derivatives vanish. The displacement vanishes at one end (say for $x = 0$) and the applied force F is known at the other end as a given function of time (for $x = l$, where l is the length of the studied body). Moreover the double forces at both ends of the body are assumed time independent. We assume that the state is homogeneous all along the one dimensional body. In this case, balance equations imply that the double forces at the ends of the studied domain and the body forces vanish. In the following, solutions of the rate problem are searched.

3.4.2. Differential equations

For this rate problem, balance equations (10) and (11) yield, respectively:

$$\dot{\sigma}' - \dot{\tau}' = 0 \tag{41}$$

$$\dot{\chi}' - \dot{\tau} = 0 \tag{42}$$

and the static boundary conditions (12) and (13) become for $x = l$:

$$\dot{\sigma} - \dot{\tau} = \dot{F} \tag{43}$$

and for $x = 0$ and for $x = l$:

$$\dot{\chi} = 0 \tag{44}$$

the constitutive equation reads in any case (for the one-mechanism theory as well as for the multi-mechanism theory, for loading as well as for unloading conditions).

$$\begin{aligned} \dot{\sigma} &= \Gamma^{11}\dot{u}' + \Gamma^{12}\dot{f} + \Gamma^{13}\dot{f}' \\ \dot{\tau} &= \Gamma^{21}\dot{u}' + \Gamma^{22}\dot{f} + \Gamma^{23}\dot{f}' \\ \dot{\chi} &= \Gamma^{31}\dot{u}' + \Gamma^{32}\dot{f} + \Gamma^{33}\dot{f}' \end{aligned} \tag{45}$$

The coefficients Γ^{ij} depending on the constitutive equation and on the loading–unloading condition of the corresponding mechanism. Substituting $\dot{\sigma}$, $\dot{\tau}$ and $\dot{\chi}$ in Eqs. (41) and (42) yields

$$(\Gamma^{11} - \Gamma^{21})\dot{u}'' + (\Gamma^{12} - \Gamma^{22})\dot{f}' + (\Gamma^{13} - \Gamma^{23})\dot{f}'' = 0 \tag{46}$$

$$\Gamma^{31}\dot{u}'' - \Gamma^{21}\dot{u}' - \Gamma^{22}\dot{f} + (\Gamma^{32} - \Gamma^{23})\dot{f}' + \Gamma^{33}\dot{f}'' = 0 \tag{47}$$

If $\Gamma^{11} - \Gamma^{21} = 0$, then Eq. (46) is a differential equation in $\dot{f}$, otherwise it allows us to compute u'' as a function of $\dot{f}'$ and $\dot{f}''$, moreover integrating this equation yields

$$(\Gamma^{11} - \Gamma^{21})\dot{u}' + (\Gamma^{12} - \Gamma^{22})\dot{f} + (\Gamma^{13} - \Gamma^{23})\dot{f}' = \dot{F} \tag{48}$$

and then u' can be computed as a function of $\dot{f}$ and $\dot{f}'$. Finally substituting $\dot{u}''$ and $\dot{u}'$ in Eq. (47) gives

$$[-\Gamma^{22}(\Gamma^{11}-\Gamma^{21})-\Gamma^{21}(\Gamma^{22}-\Gamma^{12})]\dot{f}+[(\Gamma^{11}-\Gamma^{21})(\Gamma^{32}-\Gamma^{23})-\Gamma^{21}(\Gamma^{23}-\Gamma^{13})+\Gamma^{31}(\Gamma^{22}-\Gamma^{12})]\dot{f}' + [\Gamma^{33}(\Gamma^{11}-\Gamma^{21})+\Gamma^{31}(\Gamma^{23}-\Gamma^{13})]\dot{f}''=\Gamma^{21}\dot{F} \quad (49)$$

3.4.3. Principle of solving the problem

This equation is an ordinary linear differential equation the unknown of which is $\dot{f}$. Its solutions are sums of a constant and some products of exponential and sine functions. The final result depends on the roots of the corresponding characteristic equation. Then using Eqs. (46) or (48), it is possible to get $\dot{u}''$ or $\dot{u}'$ and finally by integration $\dot{u}$ which turns out to be the sum of a linear function (with respect to x) plus once more some products of exponential and sine functions. Let us notice that the linear part of the displacement field is the classical solution of a usual first gradient medium (which means a constant strain).

As the coefficients can be different according to the loading–unloading criterion(s), the solution of a given problem is a patch of elementary solutions like the previous one, each of one being valid for a given part.

In order to solve completely our problem we have

- to ensure the continuity of $\dot{u}$, $\dot{f}$, $\dot{\sigma}-\dot{\tau}$ and $\dot{\chi}$ at the common end of two parts,
- to ensure that inside a given part the chosen coefficients Γ^{ij} are in accordance with the corresponding loading–unloading condition,
- to ensure that the boundary conditions are met.

A consequence of the previous requirements, is that the points belonging to the boundary of two parts, a loading one and a unloading one, have to enjoy a neutral loading. This remark is very helpful. In fact it is the key point of the way to get the solutions when some parts of the body are loading whereas other parts are unloading (the same point holds in the case of multi-mechanism plastic model, when different parts are in different partial or total loading conditions). This is a salient difference with solutions got so far with such constitutive equations where solutions are only available for a linear media corresponding to loading (see e.g. Vardoulakis and Sulem (1995)) and called linear comparison solid referencing to the work of Hill (1958) about classical media. Although the solutions are a little bit different, the method is the same as the one described in Chambon et al. (1998). We do not want to go further as solutions are depending on the chosen constitutive equation, but it is quite clear that these solutions involve internal lengths, because there are exponentials and sines in the solution. A more complete study is made at the end of this paper for a less simple problem but for a particular case of microstructured continua (namely a second gradient model).

3.4.4. A simple multi-mechanism case

When a multi-mechanism model like the one seen in Section 3.2 is used, this means that coefficients Γ^{ij} vanish if $i \neq j$. In this case Eqs. (46)–(49) become

$$\Gamma^{11}\dot{u}''-\Gamma^{22}\dot{f}'=0 \quad (50)$$

$$\Gamma^{22}\dot{f}+\Gamma^{33}\dot{f}''=0 \quad (51)$$

in this case according to the signs of Γ^{22} and Γ^{33} the basic solutions are either exponentials or sines. The study is then rather similar to the one described in Chambon et al. (1998), two internal lengths are introduced, one corresponding to exponential the other to sines. Such models can then clearly been used to model localization phenomena and (or) boundary layers effects.

4. Microstructured continuum with kinematic constraint: Cosserat theory

Following Germain (1973a) or Vardoulakis and Sulem (1995), Cosserat theories can be seen as simplifications of the previous general theory of media with microstructure. In fact some kinematic constraint is added to the models presented in Section 2. In the next section another kinematic constraint will be added yielding local second gradient models. It is interesting here to begin with a virtual work written in a way a little bit different from that of Eq. (7).

$$w^* = \alpha_{ij} D^*_{ij} + \tau_{ij}(f^*_{ij} - R^*_{ij}) + \chi_{ijk} h^*_{ijk} \tag{52}$$

This equation is equivalent to Eq. (7) with $\alpha_{ij} = \sigma_{ij} - \tau_{ij}$.

4.1. General Cosserat theory

In a Cosserat theory the microstrain d_{ij} is assumed to vanish. This means that $f_{ij} = r_{ij}$ and that $h_{ijk} = \partial f_{ij}/\partial x_k$ has then only nine independent components because it is antisymmetric with respect to its first two indices. The density of virtual work now reads

$$w^* = \alpha_{ij} D^*_{ij} + \tau_{ij}(r^*_{ij} - R^*_{ij}) + \chi_{ijk} h^*_{ijk} \tag{53}$$

which implies that without any loss of generality τ_{ij} is antisymmetric and χ_{ijk} is antisymmetric with respect to its first two indices. The internal virtual work becomes

$$W^{*\mathrm{i}} = \int_\Omega w^* \, \mathrm{d}v = \int_\Omega (\alpha_{ij} D^*_{ij} + \tau_{ij}(r^*_{ij} - R^*_{ij}) + \chi_{ijk} h^*_{ijk}) \, \mathrm{d}v \tag{54}$$

Let us notice that when the virtual work equation (which means as usual equating the internal and external virtual work) is applied in this case, the virtual fields have to meet the kinematic constraint. This means precisely that h^*_{ijk} derives from r^*_{ij} which is antisymmetric and which vanishes on the part of the boundaries where kinematics boundary conditions hold. Applying Eq. (13) for the particular case of Cosserat continuum implies that the prescribed T_{ij} if any has to be antisymmetric.

We do not want to go into details, as Cosserat theory is described in many papers. Our purpose is only to show that this theory can be seen as a particular case of continuum with microstructure and consequently that complete analytical solutions can be got by the method proposed in Section 3.4.3.

4.2. Flow theory of plasticity for Cosserat continuum

As Cosserat theories are part of the general framework of media with microstructure, it is straightforward to develop flow theory of plasticity for Cosserat continua. In fact the Muhlhaus–Vardoulakis Cosserat model (see e.g. Muhlhaus and Vardoulakis (1987) and Vardoulakis and Sulem (1995)) is a one mechanism flow theory of plasticity as defined in Section 3.1 and it is then possible to get analytical solutions using the method proposed in Section 3.4.3. In the following a special case of Cosserat will be considered which is a simplified case of the general case and which seems more suitable to model the physics of granular materials.

4.3. Cosserat second gradient models

We will consider the general second gradient models is Section 5. Here is studied a particular case of second gradient models which is too a particular case of Cosserat model, so it is called Cosserat second gradient model. We do not want to study this point here but one of the major difficulties of all the enhanced models studied in this paper is not only the identification of the parameters needed but even the choice of specific functions needed in the model such as yield functions, direction of the plastic strain and so on. It would be interesting to be guided by experimental data. Unfortunately needed experiments are necessarily inhomogeneous and very few of such experiments are available. Enhanced models need inverse analysis.

However for granular material we have some insight into physical phenomena. Calvetti et al. (1997) show clearly that in most cases (except for some exotic loading paths the data of which are not so clear even if the trends are similar) the macro rotation is the same as the average of the rotation of the grains (here as 2D experiments are performed, the grains are small rods). As we are dealing with continuum media and thus as the mathematical description of the media needs continuous functions, it is our opinion that the average rotation of the grains has to be interpreted as the microrotation. So for granular material, it is reasonable to assume that the macrorotation is equal to the microrotation and so we have

$$r_{ij} = R_{ij} \tag{55}$$

Then the internal virtual work becomes

$$W_i^* = \int_\Omega w^* \, \mathrm{d}v = \int_\Omega (\alpha_{ij} D_{ij}^* + \chi_{ijk} h_{ijk}^*) \, \mathrm{d}v \tag{56}$$

where $h_{ijk}^* = \partial R_{ij}^* / \partial x_k$. Application of virtual work principle and two successive integrations by part yield only one balance equation:

$$\frac{\partial \alpha_{ij}}{\partial x_j} - \frac{\partial^2 \chi_{ijk}}{\partial x_j \partial x_k} + G_i = 0 \tag{57}$$

but as previously two static boundary conditions. These models are a little bit different from the ones studied up to now.

4.4. A one-dimensional application of Cosserat second gradient model

Denoting x the space variable, we assume that the only nonvanishing displacements, denoted v, are measured along a normal to the x axis and that all the quantities depending only on x. The derivative with respect to x is denoted by a $'$. The macro strain has a unique component $\frac{1}{2}v'$. The second gradient has only two nonvanishing components the values of which are $\pm\frac{1}{2}v''$. Each of the macro stress, and the double stress have a unique component denoted respectively α and χ. The body forces are assumed to remain constant, so their time derivatives vanish. The displacement vanishes at one end (say for $x = 0$) the applied force F is known at the other end and the double forces at both ends of the body are assumed not to vary. As in the general case, we assume that the state is homogeneous all along the one-dimensional body (which implies the annulment of the double stress at the boundaries) and we look for solutions of the rate problem. We have

$$\dot{\alpha}' - \dot{\chi}'' = 0 \tag{58}$$

and for $x = l$:

$$\dot{\alpha} = \dot{F} \tag{59}$$

and for $x = 0$ and for $x = l$:

$$\dot{\chi} = 0 \tag{60}$$

As a particular case of the constitutive equation studied in Section 3 we can assume the following relations, the coefficients Γ^{ij} depending on which constitutive equation is used and whether the corresponding mechanism is a loading one or not.

$$\dot{\alpha} = \Gamma^{11}\dot{v}' + \Gamma^{12}\dot{v}''$$
$$\dot{\chi} = \Gamma^{21}\dot{v}' + \Gamma^{22}\dot{v}'' \tag{61}$$

Substituting $\dot{\alpha}$, and $\dot{\chi}$ in Eq. (58) yields:

$$\Gamma^{11}\dot{v}'' + (\Gamma^{12} - \Gamma^{21})\dot{v}''' + \Gamma^{22}\dot{v}'''' = 0 \tag{62}$$

This equation is once more an ordinary linear differential equation the unknown function of which is $\dot{v}''$. The solutions can be built like in Chambon et al. (1998).

When a multi-mechanism model like the one seen in Section 3.2 is used, or if $\Gamma^{12} = \Gamma^{21}$ according to the signs of Γ^{22} and Γ^{11} the basic solutions are either exponentials or sines. The study is then rather similar to the one described in Chambon et al. (1998), two internal lengths are introduced, one corresponding to exponential the other to sines. Such models can then clearly be used to model localization phenomena and (or) boundary layer effects. It is clear however than in the true one-dimensional case like the one discussed in Section 3.4 the Cosserat effect disappears as no rotation is involved.

5. Microstructured continuum with kinematic constraint: second gradient models

5.1. The main assumption

In the framework, we are working in, a second gradient model is a model where the microstrain is assumed to be equal to the macrostrain.

$$f_{ij} = F_{ij} \tag{63}$$

and consequently

$$f_{ij} = \frac{\partial u_i}{\partial x_j} \tag{64}$$

This main assumption can be used in two different manners. The first one is using Eqs. (63) and (64), eliminating f_{ij} in each equation, and consequently assuming that $f^*_{ij} = \partial u^*_i/\partial x_j$, this is done in Section 5.2.

Another way is keeping the equations of continuum with microstructure, simplifying them only with Eq. (63) and using Eq. (64) like a mathematical constraint. Then Lagrange multipliers are introduced, corresponding to each component in Eq. (64). They are denoted λ_{ij}. The following equations are got.

$$\int_\Omega \left[\sigma_{ij}\frac{\partial u^*_i}{\partial x_j} + \chi_{ijk}\frac{\partial f^*_{ij}}{\partial x_k} + \lambda_{ij}\left(\frac{\partial u^*_i}{\partial x_j} - f^*_{ij}\right)\right] \mathrm{d}v = \int_\Omega G_i u^*_i \,\mathrm{d}v + \int_{\partial\Omega} (t_i u^*_i + T_{ij} f^*_{ij})\,\mathrm{d}s \tag{65}$$

which hold for any kinematic admissible fields u^*_i and f^*_{ij}, and

$$\int_\Omega \lambda^*_{ij}\left(\frac{\partial u_i}{\partial x_j} - f_{ij}\right) \mathrm{d}v = 0 \tag{66}$$

which holds for any λ^*_{ij}.

This way which seems less natural than the one followed in the next section is in fact very useful especially in finite element applications because it is difficult to get shape functions meeting the necessary continuity conditions. Such a method has to be compared to the one developed to deal with incompressible materials. Eq. (64) is a kinematic condition similar to incompressibility condition and then can be used in computations following the same idea. This has been done in Matsushima et al. (2000) and will be studied extensively in a forthcoming paper (see Matsushima et al. (2001)). A two-dimensional application of this way of working in finite element method for incompressible second gradient elastic solids have been done by Shu et al. (1999).

From a physical point of view, as seen in Section 4.3, it is possible to justify the use of Cosserat second gradient models for granular materials. Cosserat second gradient models can be seen also as particular second gradient models, in some sense second gradient models are justified as well. Unfortunately, to our knowledge, there is no micromechanical study that justify the assumption done in Eq. (63). However generalizing the assumption done for granular materials, it can be reasonable to assume that cohesive geomaterials (i.e. cohesive soils like clays, rocks and concrete) can be modeled by a second gradient continuum.

5.2. Equations of a second gradient model

As a consequence of Eqs. (63) and (64), the principle of virtual work reads: for every kinematic admissible field u_i^*:

$$\int_{\Omega}\left(\sigma_{ij}D_{ij}^* + \chi_{ijk}\frac{\partial^2 u_i^*}{\partial x_j \partial x_k}\right)\mathrm{d}v = \int_{\Omega} G_i u_i^*\,\mathrm{d}v + \int_{\partial\Omega}\left(t_i u_i^* + T_{ij}\frac{\partial u_i^*}{\partial x_j}\right)\mathrm{d}s \tag{67}$$

Let us notice here that we have only one real unknown field u_i and similarly only one virtual displacement field, which is rather different from the virtual work principle applied in Section 2.4 or in Section 5.1. It is worth noticing a second point, as u_i^* and $\partial u_i^*/\partial x_j$ are not independent because the value of u_i and its tangential derivatives (along the boundary) cannot vary independently, t_i and T_{ij} cannot be taken independently. Let us denote D the normal derivative of any quantity q, $(Dq = (\partial q/\partial x_k)n_k)$ and D_j the tangential derivatives $(D_j q = \partial q/\partial x_j - (\partial q/\partial x_k)n_k n_j)$. It is more convenient to rewrite the external virtual work with p_i and P_i like in the following virtual work principle equation:

$$\int_{\Omega}\left(\sigma_{ij}D_{ij}^* + \chi_{ijk}\frac{\partial^2 u_i^*}{\partial x_j \partial x_k}\right)\mathrm{d}v = \int_{\Omega} G_i u_i^*\,\mathrm{d}v + \int_{\partial\Omega}(p_i u_i^* + P_i D u_i^*)\,\mathrm{d}s \tag{68}$$

In this case p_i and P_i can be chosen independently.

Application of virtual work principle equation (68) and two integrations by part give the balance equation and the boundary conditions. The balance equations read

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\partial^2 \chi_{ijk}}{\partial x_j \partial x_k} + G_i = 0 \tag{69}$$

These equations are identical to the ones got for the Cosserat second gradient model but here there is no symmetry condition about the double stress χ_{ijk}.

Here the boundary conditions are less simple due to the relation between u_i and $f_{ij} = \partial u_i/\partial x_j$ and consequently between the corresponding virtual quantities on the part of the boundary where the forces and double forces are prescribed. Finally as it is assumed that the boundary is regular (which means existence and uniqueness of the normal for every point of the boundary $\partial\Omega$ of the studied domain), after one more integration by parts, we get

$$\sigma_{ij}n_j - n_k n_j D\chi_{ijk} - \frac{D\chi_{ijk}}{Dx_k}n_j - \frac{D\chi_{ijk}}{Dx_j}n_k + \frac{Dn_l}{Dx_l}\chi_{ijk}n_j n_k - \frac{Dn_j}{Dx_k}\chi_{ijk} = p_i \tag{70}$$

and

$$\chi_{ijk}n_j n_k = P_i \tag{71}$$

where p_i and P_i are prescribed.

5.3. Local elasto-plastic second gradient models

It is now easy to develop an elasto-plastic flow theory of plasticity in the spirit of the work done in Section 3. The generalized stresses Σ are now σ_{ij} and χ_{ijk} and the generalized strains E are D_{ij} and h_{ijk}. However there is no theoretical reason to assume a link between the plastic part of D_{ij} and the one of h_{ijk}. If such a link is assumed then we end up with a nonlocal model. For instance the model of Vardoulakis and Frantziskonis (1992) and Frantziskonis and Vardoulakis (1992) is not a local elasto-plastic second gradient model because, as mentioned in Section 1, in this case the constitutive equation is itself a partial differential equation.

5.3.1. The Fleck Hutchinson model

In fact a theory within this framework has already been develop by Fleck and Hutchinson for metals (see Fleck and Hutchinson (1997)). This model can be called a one mechanism plastic second gradient model because there is only one yield surface which is a generalization accounting for second gradient terms of the classical (in metal plasticity) J^2 flow theory of plasticity. Moreover the authors assume that the plastic flow is normal to the yield surface which is usual for metal plasticity but is irrelevant for geomaterials. Using this assumption and working in the general framework of local plastic second gradient model allow the authors to extend the classical minimum principles to second gradient models. However the analytical solutions they gave are not completely satisfactory as they do not use the constraints depicted in Section 3.4.3 especially the neutral loading condition at a common end of a loading part and an unloading one.

5.3.2. General framework of local plane elasto-plastic second gradient models

In this section we intend to give an example of elasto-plastic second gradient model. Some simplifications are assumed keeping however the main features of the model.

It is assumed that the elastic part of the model is isotropic and linear. In this case as proved by Mindlin (see e.g. Mindlin (1964, 1965)), we have

$$\dot{\sigma}_{ij} = \Gamma^1_{ijkl}\dot{D}^{\mathrm{e}}_{kl} \tag{72}$$

$$\dot{\chi}_{ijk} = \Gamma^2_{ijklmn}\dot{h}^{\mathrm{e}}_{lmn} \tag{73}$$

where Γ^1_{ijkl} depends as usual on two different parameters and Γ^2_{ijklmn} depends on five different parameters.

We assume that the yield function depends only on σ_{ij}. Moreover we assume that $\dot{h}^{\mathrm{p}}_{lmn} = 0$. And thus the model can be written

$$\dot{\chi}_{ijk} = \Gamma^2_{ijklmn}\dot{h}_{lmn} \tag{74}$$

for loading and unloading, and

$$\dot{\sigma}_{ij} = \Gamma^1_{ijkl}\dot{D}_{kl} \tag{75}$$

for unloading only and

$$\dot{\sigma}_{ij} = \Gamma^1_{ijkl}\dot{D}_{kl} - \frac{\Gamma^1_{ijpq}\psi^D_{pq}\frac{\partial\phi}{\partial\sigma_{mn}}\Gamma^1_{mnkl}\dot{D}_{kl}}{h^\sigma + \frac{\partial\phi}{\partial\sigma_{mn}}\Gamma^1_{mnkl}\psi^D_{kl}} \tag{76}$$

for loading only.

In order to clarify the following we deal now only with two-dimensional problems. In this case Eq. (74) can be rewritten in a matrix form.

$$\begin{bmatrix} \dot{\chi}_{111} \\ \dot{\chi}_{112} \\ \dot{\chi}_{121} \\ \dot{\chi}_{122} \\ \dot{\chi}_{211} \\ \dot{\chi}_{212} \\ \dot{\chi}_{221} \\ \dot{\chi}_{222} \end{bmatrix} = \begin{bmatrix} a^{12345} & 0 & 0 & a^{23} & 0 & a^{12} & a^{12} & 0 \\ 0 & a^{145} & a^{145} & 0 & a^{25} & 0 & 0 & a^{12} \\ 0 & a^{145} & a^{145} & 0 & a^{25} & 0 & 0 & a^{12} \\ a^{23} & 0 & 0 & a^{34} & 0 & a^{25} & a^{25} & 0 \\ 0 & a^{25} & a^{25} & 0 & a^{34} & 0 & 0 & a^{23} \\ a^{12} & 0 & 0 & a^{25} & 0 & a^{145} & a^{145} & 0 \\ a^{12} & 0 & 0 & a^{25} & 0 & a^{145} & a^{145} & 0 \\ 0 & a^{12} & a^{12} & 0 & a^{23} & 0 & 0 & a^{12345} \end{bmatrix} \begin{bmatrix} \dot{h}_{111} \\ \dot{h}_{112} \\ \dot{h}_{121} \\ \dot{h}_{122} \\ \dot{h}_{211} \\ \dot{h}_{212} \\ \dot{h}_{221} \\ \dot{h}_{222} \end{bmatrix} \quad (77)$$

where, all the terms depend on the five constants a^1, a^2, a^3, a^4, a^5 defined by Mindlin (see e.g. Mindlin (1965, 1964) according to the following formulae:

$$\begin{aligned} a^{12345} &= 2(a^1 + a^2 + a^3 + a^4 + a^5) \\ a^{23} &= a^2 + 2a^3 \\ a^{12} &= a^1 + a^2/2 \\ a^{145} &= a^1/2 + a^4 + a^5/2 \\ a^{25} &= a^2/2 + a^5 \\ a^{34} &= 2(a^3 + 2a^4) \end{aligned} \quad (78)$$

5.3.3. *Cosserat second gradient models revisited*

Cosserat second gradient models can be seen as degenerate cases of second gradient models. However the way of getting this Cosserat second gradient model from general second gradient models is not straightforward. Such a derivation needs somewhat long mathematical manipulations and will not be made in this paper. The difficulties are similar (but concerning second gradient terms) to the ones which arise when we have to deal with incompressible media seen as particular cases of classical first gradient models.

5.3.4. *A plane Mohr–Coulomb second gradient models*

In the following it is assumed that the yield function and the direction of the plastic strain are of Mohr–Coulomb type. Consequently, using for the classical part of the model, the 2D Mohr–Coulomb constitutive equations of Vardoulakis and Sulem (see Vardoulakis and Sulem (1995), chapter 6), yields the following model. The second gradient part obeys Eq. (77). The more classical part can be summarized by the following equations. For unloading we have

$$\begin{bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \\ \dot{\sigma}_{21} \end{bmatrix} = \begin{bmatrix} K+G & K-G & 0 & 0 \\ K-G & K+G & 0 & 0 \\ 0 & 0 & G & G \\ 0 & 0 & G & G \end{bmatrix} \begin{bmatrix} \dot{D}_{11} \\ \dot{D}_{22} \\ \dot{D}_{12} \\ \dot{D}_{12} \end{bmatrix} \quad (79)$$

where K and G are, respectively, the bulk modulus and the shear modulus. In the following, Γ^1_{ijkl} denotes as well elastic moduli as elasto-plastic moduli, according to the context. This is not ambiguous, for instance in Eq. (80), it is clear that Γ^1_{ijkl} denotes elasto-plastic moduli. And for loading, Eq. (76) holds for $i, j \in \{1, 2\}$ and we have:

$$
\begin{aligned}
\Gamma^1_{1111} &= K + G - \frac{1}{H}\left(K\mu + G\frac{\sigma_{11}-\sigma_{22}}{2\tau}\right)\left(K\beta + G\frac{\sigma_{11}-\sigma_{22}}{2\tau}\right)\\
\Gamma^1_{1112} &= \Gamma^1_{1121} = -\frac{1}{H}\left(K\beta + G\frac{\sigma_{11}-\sigma_{22}}{2\tau}\right)\left(\frac{G\sigma_{12}}{\tau}\right)\\
\Gamma^1_{1122} &= K - G - \frac{1}{H}\left(K\beta + G\frac{\sigma_{11}-\sigma_{22}}{2\tau}\right)\left(K\mu + G\frac{\sigma_{22}-\sigma_{11}}{2\tau}\right)\\
\Gamma^1_{1211} &= \Gamma^1_{2111} = -\frac{1}{H}\left(\frac{G\sigma_{12}}{\tau}\right)\left(K\mu + G\frac{\sigma_{11}-\sigma_{22}}{2\tau}\right)\\
\Gamma^1_{1212} &= \Gamma^1_{1221} = \Gamma^1_{2112} = \Gamma^1_{2121} = G - \frac{1}{H}\left(\frac{G\sigma_{12}}{\tau}\right)^2\\
\Gamma^1_{1222} &= \Gamma^1_{2122} = -\frac{1}{H}\left(\frac{G\sigma_{12}}{\tau}\right)\left(K\mu + G\frac{\sigma_{22}-\sigma_{11}}{2\tau}\right)\\
\Gamma^1_{2211} &= K - G - \frac{1}{H}\left(K\mu + G\frac{\sigma_{22}-\sigma_{11}}{2\tau}\right)\left(K\beta + G\frac{\sigma_{11}-\sigma_{22}}{2\tau}\right)\\
\Gamma^1_{2212} &= \Gamma^1_{2221} = -\frac{1}{H}\left(K\beta + G\frac{\sigma_{22}-\sigma_{11}}{2\tau}\right)\left(\frac{G\sigma_{12}}{\tau}\right)\\
\Gamma^1_{2222} &= K + G - \frac{1}{H}\left(K\mu + G\frac{\sigma_{22}-\sigma_{11}}{2\tau}\right)\left(K\beta + G\frac{\sigma_{22}-\sigma_{11}}{2\tau}\right)
\end{aligned}
\tag{80}
$$

where σ_{ij} are the stress, τ is the stress deviator defined by

$$
\tau = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + (\sigma_{12})^2} \tag{81}
$$

and μ and β are, respectively, the friction angle and the dilation angle. H is defined by the following equation:

$$
H = h + K\mu\beta + G \tag{82}
$$

where h is as usual the hardening modulus.

6. An application of local elasto-plastic second gradient model

6.1. The problem to be solved

In this section solutions for a boundary value problem are derived. We consider an infinite layer of geomaterials bounded by two parallel plane corresponding to $x = 0$ and $x = l$ (see Fig. 1). z is the direction normal to the studied part of plane. This means that plane strains are assumed in this direction. The velocity field is defined by only two variables namely u in the x direction and v in the y direction. Moreover it is assumed that u and v are functions of x only. So in the following derivatives with respect to x are denoted by a $'$ which is here unambiguous. The model defined in Section 5.3.2 is used. That is a second gradient model with only one mechanism and such that $\dot{h}^p_{lmn} = 0$. We assume that the present stress state is homogeneous and we search solutions of the rate problem. As already mentioned in Section 3.4.1, this means that boundary conditions and body forces history allow for an homogeneous solution. The rates of boundary conditions are known namely $\dot{u} = 0$, $\dot{v} = 0$ and $\dot{P}_i = 0$ for $x = 0$ and $\dot{p}_i$ and $\dot{P}_i$ are given for $x = l$. Other boundary conditions can be chosen provided a sufficient number of boundary conditions corresponding to the second gradient term are considered. The body forces are assumed to be constant.

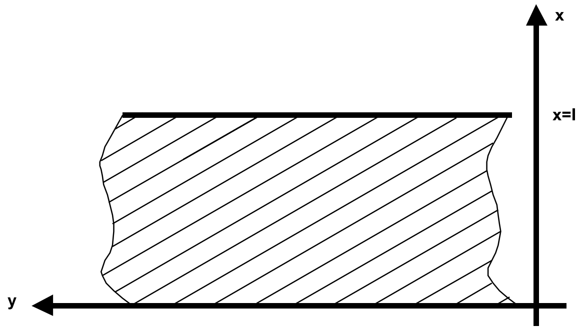

Fig. 1. Studied domain.

We have

$$\left[\dot{F}\right] = \begin{bmatrix} \dot{u}' & 0 \\ \dot{v}' & 0 \end{bmatrix} \tag{83}$$

$$\left[\dot{D}\right] = \begin{bmatrix} \dot{u}' & \frac{1}{2}\dot{v}' \\ \frac{1}{2}\dot{v}' & 0 \end{bmatrix} \tag{84}$$

$$\left[\dot{R}\right] = \begin{bmatrix} 0 & -\frac{1}{2}\dot{v}' \\ \frac{1}{2}\dot{v}' & 0 \end{bmatrix} \tag{85}$$

The only nonvanishing components of h_{ijk} are

$$\dot{h}_{111} = \dot{u}'' \tag{86}$$

and

$$\dot{h}_{211} = \dot{v}'' \tag{87}$$

The balance equation (69) read

$$\dot{\sigma}'_{11} - \dot{\chi}''_{111} = 0 \tag{88}$$

and

$$\dot{\sigma}'_{21} - \dot{\chi}''_{211} = 0. \tag{89}$$

The boundary conditions (Eq. (70)) become for $x = l$:

$$\dot{\sigma}_{11} - \dot{\chi}'_{111} = \dot{p}_1 \;\; \dot{\sigma}_{21} - \dot{\chi}'_{211} = \dot{p}_2 \tag{90}$$

and

$$\dot{\chi}_{111} = \dot{P}_1 \;\; \dot{\chi}_{211} = \dot{P}_2. \tag{91}$$

Taking into account the boundary conditions, the balance equations can be integrated once, which yields

$$\dot{\sigma}_{11} - \dot{\chi}'_{111} = \dot{p}_1 \tag{92}$$

and

$$\dot{\sigma}_{21} - \dot{\chi}'_{211} = \dot{p}_2 \tag{93}$$

Using now the constitutive equations yields

$$\Gamma^1_{1111}\dot{u}' + \Gamma^1_{1112}\dot{v}' - a^{12345}\dot{u}''' = \dot{p}_1 \tag{94}$$

and

$$\Gamma^1_{2111}\dot{u}' + \Gamma^1_{2121}\dot{v}' - a^{34}\dot{v}''' = \dot{p}_2 \tag{95}$$

6.2. Partial solutions

Solutions of such a set of equations depend on the values of the constitutive coefficients. These values depend on the stress value. In fact as we choose to work with isotropic models, only the stress orientation influences the values of the plastic constitutive coefficients. For a given material, apart from the boundary conditions, the solution depends only on the orientation of the stress tensor with respect to the chosen axis. Our problem can be seen as a general shear band analysis where the shear band orientation is assumed but the stress orientation is free. It is a shear band analysis in the spirit of Vermeer (1982) but for an elasto-plastic second gradient model.

Solutions of Eqs. (94) and (95) are depending on the roots of the characteristic equation (96).

$$A(s)^4 - B(s)^2 + C = 0 \tag{96}$$

where

$$\begin{aligned} A &= a^{12345}a^{34} \\ B &= a^{12345}\Gamma^1_{2121} + a^{34}\Gamma^1_{1111} \\ C &= \Gamma^1_{2121}\Gamma^1_{1111} - \Gamma^1_{1112}\Gamma^1_{2111} \end{aligned} \tag{97}$$

Denoting $S = (s)^2$, Eq. (96) reads

$$A(S)^2 - BS + C = 0 \tag{98}$$

It is worth noticing that Δ the discriminant of Eq. (98) reads

$$\Delta = (a^{12345}\Gamma^1_{2121} - a^{34}\Gamma^1_{1111})^2 + 4a^{12345}a^{34}\Gamma^1_{1112}\Gamma^1_{2111} \tag{99}$$

which is positive for elastic moduli and also for elasto-plastic moduli corresponding to materials obeying the normality rule. For other elasto-plastic materials as C is usually decreasing as the material is loaded (see Section 6.4), Δ is positive and the roots of Eq. (98) are necessarily real.

Finally, the solutions of Eqs. (94) and (95) have the following forms:

(1) If the two roots of Eq. (98) are positive, then the solutions read

$$\begin{bmatrix} \dot{u}' \\ \dot{v}' \end{bmatrix} = \begin{bmatrix} \dot{U}'^0 \\ \dot{V}'^0 \end{bmatrix} + \begin{bmatrix} \dot{U}'^1 \\ \dot{V}'^1 \end{bmatrix}(\lambda^{11}\cosh(\eta^1 x) + \lambda^{12}\sinh(\eta^1 x)) + \begin{bmatrix} \dot{U}'^2 \\ \dot{V}'^2 \end{bmatrix}(\lambda^{21}\cosh(\eta^2 x) + \lambda^{22}\sinh(\eta^2 x)) \tag{100}$$

where $\dot{U}'^0$ and $\dot{V}'^0$ are solutions of equation:

$$\begin{bmatrix} \Gamma^1_{1111} & \Gamma^1_{1112} \\ \Gamma^1_{2111} & \Gamma^1_{2121} \end{bmatrix} \begin{bmatrix} \dot{U}'^0 \\ \dot{V}'^0 \end{bmatrix} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} \tag{101}$$

and $(\eta^1)^2 = S^1$ and $(\eta^2)^2 = S^2$ are the two roots of Eq. (98). Moreover $\dot{U}'^i$ and $\dot{V}'^i$, $i \in \{1, 2\}$ have to meet:

$$\begin{bmatrix} \Gamma^1_{1111} - S^i a^{12345} & \Gamma^1_{1112} \\ \Gamma^1_{2111} & \Gamma^1_{2121} - S^i a^{34} \end{bmatrix} \begin{bmatrix} \dot{U}'^i \\ \dot{V}'^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{102}$$

(2) If one root of Eq. (98) denoted S^1 is positive and the other one denoted S^2 is negative, then the solutions read

$$\begin{bmatrix} \dot{u}' \\ \dot{v}' \end{bmatrix} = \begin{bmatrix} \dot{U}'^0 \\ \dot{V}'^0 \end{bmatrix} + \begin{bmatrix} \dot{U}'^1 \\ \dot{V}'^1 \end{bmatrix} (\lambda^{11}\cosh(\eta^1 x) + \lambda^{12}\sinh(\eta^1 x)) + \begin{bmatrix} \dot{U}'^2 \\ \dot{V}'^2 \end{bmatrix} (\mu^{21}\cos(\omega^2 x) + \mu^{22}\sin(\omega^2 x)) \tag{103}$$

where $\dot{U}'^0$ and $\dot{V}'^0$ are solutions of Eq. (101). We have $(\eta^1)^2 = S^1$ and $(\omega^2)^2 = -S^2$ and $\dot{U}'^i$ and $\dot{V}'^i$, $i \in \{1, 2\}$ have to meet Eq. (102).

(3) If one root of Eq. (98) vanishes which implies that $C = 0$ and the other denoted S^1 is positive, then the solutions read

$$\begin{aligned} \begin{bmatrix} \dot{u}' \\ \dot{v}' \end{bmatrix} = {} & \begin{bmatrix} \dot{U}'^1 \\ \dot{V}'^1 \end{bmatrix} (\lambda^{11}\cosh(\eta^1 x) + \lambda^{12}\sinh(\eta^1 x)) + \begin{bmatrix} \Gamma^1_{1112}\dot{p}_2 - \Gamma^1_{2121}\dot{p}_1 \\ \Gamma^1_{2111}\dot{p}_1 - \Gamma^1_{1111}\dot{p}_2 \end{bmatrix} \frac{(x)^2}{2B} + \begin{bmatrix} -\Gamma^1_{1112} \\ \Gamma^1_{1111} \end{bmatrix} k^1 \frac{x}{B} \\ & + \begin{bmatrix} \frac{a^{12345}\Gamma^1_{1112}\dot{p}_2 + a^{34}\Gamma^1_{1111}\dot{p}_1}{B(\Gamma^1_{1111} + \Gamma^1_{1112})} + \frac{B\Gamma^1_{1112}}{a^{12345}\Gamma^1_{1112}\dot{p}_2 + a^{34}\Gamma^1_{1111}\dot{p}_1} k^2 \\ \frac{a^{12345}\Gamma^1_{1112}\dot{p}_2 + a^{34}\Gamma^1_{1111}\dot{p}_1}{B(\Gamma^1_{1111} + \Gamma^1_{1112})} - \frac{B\Gamma^1_{1111}}{a^{12345}\Gamma^1_{1112}\dot{p}_2 + a^{34}\Gamma^1_{1111}\dot{p}_1} k^2 \end{bmatrix} \end{aligned} \tag{104}$$

where we have $(\eta^1)^2 = S^1$; $\dot{U}'^i$ and $\dot{V}'^i$, $i \in \{1, 2\}$ have to meet Eq. (102).

It is not necessary to study other solutions for the characteristic equation, because as it will be discussed in Section 6.4, C is the localization criterion of the underlying classical elasto-plastic medium and it is not likely that we have both C positive and localization. This excludes the possibility of two negative roots for the characteristic equation.

6.3. Patch conditions and full solutions

Given a stress state, depending on whether loading or unloading condition is assumed, two partial solutions are possible. A part of the body where the loading solution is used is called a soft part. Similarly, a part of the body where unloading solution is used is called a hard part. Given a stress state, there are two different partial possible solutions. Each solution corresponds to formulae chosen among the three possible ones depicted in the previous section, depending on the value of the constitutive parameters, on the stress and on the orientation of the stress with respect to the chosen axes. Let us notice that each partial solution depends on four independent constants.

The solution of a given problem is then a patch of partial solutions. The body is split in hard parts or soft parts. Let us denote N the number of parts. To get a complete solution, it is necessary to find five unknowns per part, four ones corresponding to the independent constants and the fifth one to the length of the part. Globally, we have $5N$ unknowns. At each of the $N - 1$ junction points it is necessary to write the continuity of $\dot{u}'$ and $\dot{v}'$. Moreover as a junction point belongs to a hard part as well as to a soft part, at this junction point the strains have to correspond to neutral loading. In the case of the particular model described in Section 5.3.4 and used to find the solutions presented in Section 6.5 the neutral loading is characterized by Eq. (105).

$$(\sigma_{11} - \sigma_{22})G\dot{u}' + 2\mu\tau K\dot{u}' + 2\sigma_{12}G\dot{v}' = 0 \tag{105}$$

It is necessary to write the balance equation at a junction point. The ones corresponding to the classical term are necessarily met as Eqs. (88) and (89) have been integrated into Eq. (90). On the contrary continuity of $\dot{\chi}_{111}$ and $\dot{\chi}_{211}$ at a junction point has to be enforced. Finally every junction point gives us five equations, which mean an amount of $5(N-1)$ equations.

Moreover it is necessary to write the boundary conditions on the two sides of the body which means that $\dot{\chi}_{111}$ and $\dot{\chi}_{211}$ are known for $x=0$ where they vanish and for $x=l$ where they are given by Eq. (91). Finally the sum of the width of the part has to be equal to l. We end up with a system of $5N$ equations of $5N$ unknowns which generally give us $\dot{u}'$ and $\dot{v}'$ as a function of x. It is then necessary to check that unloading condition holds in hard part whereas loading condition holds in soft parts. At the end it is possible to get $\dot{u}$ and $\dot{v}$ as we know that for $x=0$: $\dot{u}=0$ and $\dot{v}=0$.

This is the generalization of the method given in Chambon et al. (1998) for a one-dimensional case. We can build up basic solutions for the rate problem of a local elasto plastic second gradient model. This is useful to check the numerical codes. It is also interesting to discuss a little bit the conditions of apparition of the diverse solutions depicted in Section 6.2.

6.4. Discussion

It is worth noticing first that as they correspond to an elastic behavior, the two parameters a^{12345} and a^{34} are positive, so $A>0$ (Eq. (98)) is always positive and C has the same sign as the product of the two roots of equation (98). Moreover B has the opposite sign of the two roots sum.

An other important remark is that in fact we can write:

$$C = \det(n_i \Gamma^1_{ijkl} n_l) \tag{106}$$

where n_i is the normal to the chosen boundaries (here $n_1 = 1$ and $n_2 = 0$). This means that C is in fact the determinant of the acoustic tensor of the underlying first gradient model, involved in the classical Rice shear band analysis (see e.g. Rice (1976)). This is why we did not study the possibility of two negative roots for the characteristic equation in Section 6.2. If our medium behaves elastically then $C>0$ and $B>0$. In this case the only partial solution is solution 1. In this case, $\Gamma^1_{1112} = \Gamma^1_{2111} = 0$, the two roots of the characteristic equation are positive and we get: $\eta^1 = \Gamma^1_{2121}/a^{34}$ and $\eta^2 = \Gamma^1_{1111}/a^{12345}$. Finally there is no coupling between the two differential equations and the solutions can be written:

$$\dot{u}' = \dot{U}'^0 + \lambda^{11}\cosh(\eta^1 x) + \lambda^{12}\sinh(\eta^1 x) \tag{107}$$

$$\dot{v}' = \dot{V}'^0 + \lambda^{21}\cosh(\eta^2 x) + \lambda^{22}\sinh(\eta^2 x) \tag{108}$$

The first terms of Eqs. (107) and (108) describe a homogeneous straining of the whole body.

For moderate plastic parts (which means very high values of h) the behavior is similar to an elastic one, and only parts obeying Eq. (100) are available.

Let us examine now what happens if the medium behaves more and more plastically. First one of the root of Eq. (98) vanishes while the other one remains positive. In this case, unloading parts obeying Eq. (104) become available. Then this root becomes negative and solution (103) corresponds to unloading parts of the body. As this solution involves a cosine it is clear that a localized structure appears. This structure has a clear internal length $2\pi/\omega^2$. It can be concluded first that the threshold of possible localization in a second gradient model like the one studied here is the same as the threshold of localization for the corresponding underlying first gradient model. It is important to emphasize that the localization criterion of studied second gradient models involves only the first gradient part of the model. However, as localized solutions are linked with a characteristic length which decreases as the medium is more and more plastic and which is infinite at the threshold of localization the appearance of a localized band depends on the geometry and on the boundary conditions of the whole problem. Practically this means that appearance of localization can be somewhat delayed for a second gradient media.

If we are dealing with a material for which normality holds and if the chosen axes are the principal directions of the stress then $\Gamma^1_{1112} = \Gamma^1_{2111} = 0$ and similarly to elastic solutions, equations for $\dot{v}'$ and $\dot{v}'$ are uncoupled.

Keeping now the same value for the stress and the hardening parameter h, but modifying now the orientation of the stress with respect to the chosen axes (which means that the components σ_{11}, σ_{22} and σ_{12} vary according to the classical formulae or rotation of a second order tensor) allow us to exhibit additional remarks. For every case seen above the solution of the characteristic equation varies continuously with respect to the orientation of the stress. Consequently as for the classical model (i.e. first gradient model) localization can be possible for a fan of stress orientation (with respect to the shear band). However once more for the second gradient model only, it is possible that details in the geometry of the body and/or of the boundary conditions inhibit the localization phenomenon.

6.5. Examples

In order to test the feasibility of the proposed way of solving some rate problems for local second gradient models, we present hereafter two solutions. Both are built up by assuming a central loading (soft) zone surrounded by two unloading (hard) zones (of course, other solutions are possible). The model used is the one detailed in Section 5.3.4. The nonlinear equations described in Section 6.3 are solved in an iterative manner.

In both computations, the constitutive parameters are

- $K = 50$ MPa
- $G = 62.5$ MPa
- $\beta = 0.25$ rad

The values of the parameters are chosen to be reasonably consistent with the Vardoulakis example (see Vardoulakis and Sulem (1995), chapter 6). The stress state is the same in both cases but with a different orientation with respect to x axis. The principal stress values are -1.8 MPa and -0.3 MPa, as we use the classical sign convention, they are negative which means that they are compression stresses. This gives a mobilized friction angle $\mu = 0.714$ rad. In case one x and y are the principal directions.

- $\sigma_{xx} = -1.8$ MPa
- $\sigma_{yy} = -0.3$ MPa
- $\sigma_{xy} = 0$ MPa

In the second case the angle between the x axis and the principal direction is -0.8 rad, which gives the following values

- $\sigma_{xx} = -1.028$ MPa
- $\sigma_{yy} = -1.072$ MPa
- $\sigma_{xy} = -0.750$ MPa

The width of the studied domain l is chosen to be 15 cm. a^{12345} and a^{34} are chosen to give reasonable internal lengths for a sand.

- $a^{12345} = 25$ MPa m
- $a^{34} = 2.5$ MPa m

Finally, besides the orientation of the stress state with respect to the x axis, the boundary conditions differ. In case one we have

- $\dot{t}_x = 1$
- $\dot{t}_y = 0$

In case two we have:

- $\dot{t}_x = 0$
- $\dot{t}_y = 1$

The units are not specified because the time unit is not specified in this kind of problem.

Let us now discuss the results. It is clear that the case number one is in fact a one-dimensional problem as the x axis is a symmetry one whereas case 2 is a true 2D problem. In both cases a deformed grid computed by multiplying the rate by a magnification factor is plotted in Fig. 2 for case 1 and in Fig. 4 for case 2. In order to check the loading–unloading criterion (Eq. (105)), this criterion is plotted as a function of x, in Fig. 3 for case 1 and in Fig. 5 for case 2. This allows us to clearly detect the width of the central (soft) part which are delimited by neutral loading, which means the annulment of the criterion. A comparison between Figs. 3 and 5 show clearly a great difference between the two cases whereas it is not the case by inspecting the deformed grids. This induces us to be cautious by deducing internal length by observing experimental data.

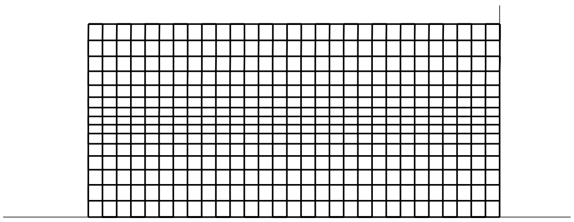

Fig. 2. Deformation pattern for case 1.

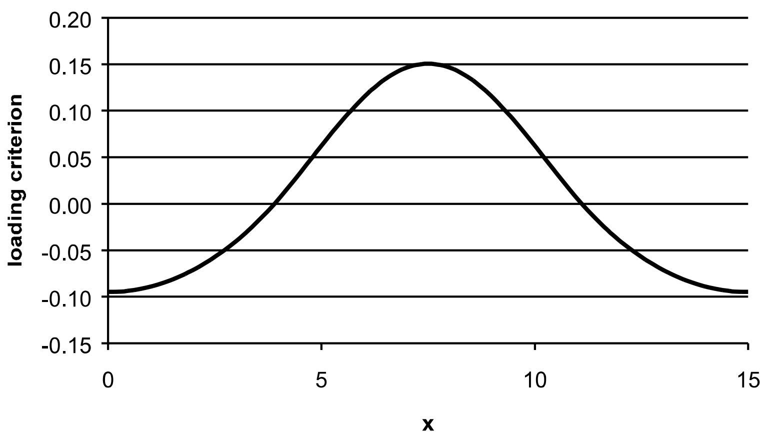

Fig. 3. Loading criterion as a function of x for case 1.

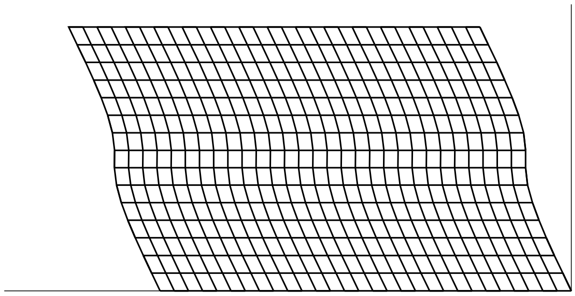

Fig. 4. Deformation pattern for case 2.

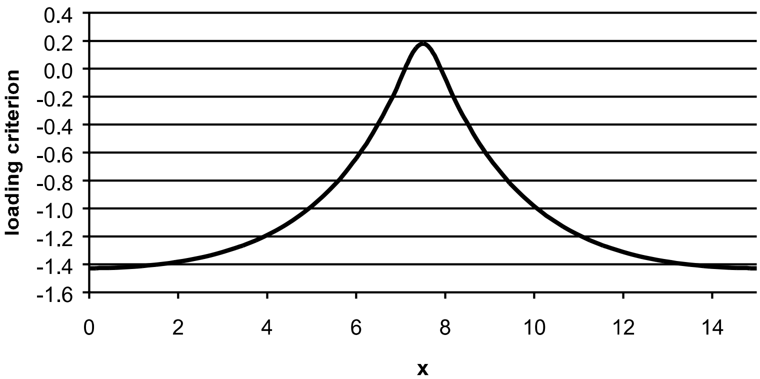

Fig. 5. Loading criterion as a function of x for case 2.

7. Conclusion

It is possible to develop straightforwardly models with microstructure inside the classical framework of plasticity by restricting such development to local models. The advantage of such studies is clear thermodynamic statements. Moreover by studying some simple boundary value rate problems, it is possible to exhibit some analytical solutions and to get some insight in shear band analysis. It is clear that all these models exhibit the same kind of behavior when the localization threshold is reached. They provide internal lengths and consequently regularization of the solutions but this does not restore uniqueness properties for the corresponding boundary value problems.

Among these models, the second gradient model and the Cosserat second gradient model are likely well adapted to geomaterials. Particularly the Cosserat second gradient model can be used for noncohesive geomaterials. Some results seen in the literature support such a conclusion. However, contrary to metal plasticity for which it is possible to base second gradient models on micro and mesoscale analysis, the micro–macro studies of geomaterials behavior are less advanced. It is likely that the good constitutive equation with microstructure useful to model geomaterials depends on the materials itself and on the pressure range. Further studies are needed.

Meanwhile, it is possible to construct solutions (even in a two-dimensional simplified case) for some rate boundary value problem involving all the models with microstructure. These problems and the corresponding solutions are clearly nonlinear as they involve loading as well unloading branches of the models. However, the only way to get general solutions for second gradient boundary value problems is the numerical computation, but knowledge of analytical solutions is a strong advantage as these solutions can be used as benchmarks for the tests of numerical codes.

A way of dealing with such models in numerical analysis by using Lagrange multipliers is suggested. Work is now in progress concerning two-dimensional finite element computations with models described in this paper.

Let us emphasize as a final comment that the method used to get the analytical solutions can be developed as in the given example for all the microstructured models. Moreover it is possible to develop such theories and such analytical studies within the damage mechanics framework in the same spirit using the same key point, i.e. the neutral loading of the junction points between loading and unloading parts of the sample.

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